

Global Invertibility and Implicit Function Theorems by Mountain Pass Theorem

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Abstract

We formulate some global invertibility and implicit function theorems. We extend the result of Idczak, Skowron and Walczak [8] on the invertibility of the operators to the case of the operators with critical points. The proof relies on the Mountain Pass Theorem combined with the Palais-Smale condition guaranteeing the claim by the invertibility of the first or the third derivative. I. e. how to solve $x^3 = y$?

1 Global Invertibility Theorem

Definition 1.1 The functional $\varphi : X \rightarrow \mathbb{R}$ is satisfying P-S, i.e. Palais-Smale, condition if any sequence x_n with $n \in \mathbb{N}$ such that $\varphi(x_n)$ is bounded and $\varphi'(x_n) \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence in the space X .

Theorem 1.1 *Let X be a Banach and H be a Hilbert space, $F \in C^3(X, H)$.*

- (a) *for any $x \in X$ either $F'(x)$ is bijective or the conjunction of conditions holds:
 $F'(x) = 0, F''(x) = 0, \sup_{|h|=1} |F'''(x)h^3| > 0$ and $F'''(x) : X^3 \rightarrow H$ is onto,*
- (b) *for any $y \in X$ the functional $\varphi_y(x) = \frac{1}{2}|F(x) - y|^2$ satisfies the P-S condition.*

Then the operator F is a globally invertible.

Proof. First, we shall prove that the operator F maps X onto H , i.e. it is surjective. To this end let us, for any fixed $y \in X$, consider the functional

$$\varphi_y(x) = \frac{1}{2}|F(x) - y|^2. \quad (1.1)$$

Since, by (b), P-S holds for φ_y , there exists a minimizer x_* for φ_y , being therefore a critical point for φ_y and satisfying thus

$$\varphi'_y(x_*) = 0. \quad (1.2)$$

The case when $F'(x_*)$ is bijective was treated extensively in [8] and as the idea is similar to the second case presented below we omit it. Let us suppose therefore that the second part of condition (a) holds, i.e. $F'(x_*) = F''(x_*) = 0$. Then we calculate the derivatives

$$\varphi'_y(x)h = \langle F(x) - y, F'(x)h \rangle, \quad (1.3)$$

$$\varphi''_y(x)h^2 = |F'(x)h|^2 + \langle F(x) - y, F''(x)h^2 \rangle, \quad (1.4)$$

$$\varphi'''_y(x)h^3 = 3 \langle F'(x)h, F''(x)h^2 \rangle + \langle F(x) - y, F'''(x)h^3 \rangle. \quad (1.5)$$

Using the Taylor expansion for the function φ_y in the neighbourhood of x_* we get for any $x \in X$ that

$$\varphi_y(x) - \varphi_y(x_*) = \varphi'''_y(x_*)(x - x_*)^3 + o(|x - x_*|^3). \quad (1.6)$$

Note that this implies that necessarily $\varphi'''_y(x_*) = 0$ since otherwise the left hand side of (1.6) would be nonnegative while the right hand side would change the sign for $|x - x_*|$ small enough if we set $x - x_* = h$ or $x - x_* = -h$ for some $h \neq 0$ small enough. But then from the last formula in (1.3) and the last part of the assumption (a) stating that $F'''(x_*)$ is onto follows that $F(x_*) - y = 0$ which ends this part of the proof, i.e. that F is onto.

The injectivity of F follows directly from assumption (a) in a standard argument following [8] using the Mountain Pass Theorem and the Taylor expansion of the function F . Indeed, suppose, on the contrary that F is not injective, i.e. there exist $x_1 \neq x_2$ such that $F(x_1) = F(x_2)$. Define

$$\psi(x) = \frac{1}{2}|F(x + x_1) - F(x_2)|^2.$$

Notice that ψ is a C^1 mapping, it enjoys Palais-Smale property and satisfies the assumptions of the Mountain Pass Theorem with $e = x_2 - x_1$ and $\alpha = \frac{1}{8}\alpha_{x_1}^2\rho^2$ where $0 < \rho < |x_2 - x_1|^2$ and $|F'(x_1)x| \geq \alpha_{x_1}|x|$ or $|F'''(x_1)x^3| \geq \alpha_{x_1}|x|^3$. Using the Taylor formula up to the first or the third order one gets for any small $|x|$ that

$$\psi(x) \geq \frac{1}{2} \left(1 - \frac{1}{2}\right)^2 \alpha_{x_1}^2 |x|^6.$$

Note that by the use of the Mountain Pass Theorem the existence of the critical point x_* for ψ follows which implies the claim due to one of the following equalities

$$\begin{aligned} 0 &= \psi'(x_*)h = \langle F(x_* + x_1) - F(x_2), F'(x_* + x_1)h \rangle, \\ 0 &= \psi'''(x_*)h^3 = \langle F(x_* + x_1) - F(x_2), F'''(x_* + x_1)h^3 \rangle. \end{aligned}$$

The latter equality is true due to the general deformation Lemma suggested by Brezis and proved Shafrir [14] while presented as Thm. 4.7 in the book of Jabri “The Mountain Pass Theorem: Variants, Generalizations and Some Applications” [10] Then the proof follows as in the Mountain Pass Theorem following from the standard deformation lemma. To this end we take the approximating curve almost realizing the critical value c from in the Mountain Pass Theorem, i.e. such that its maximum value lies in $(c, c + \varepsilon]$ for arbitrarily small value ε . Then due to the Deformation Lemma by Shafrir [14] we can deform it with η while decreasing the value of the functional ψ , in the direction defined by assumption (b), below the critical value $c = \inf_{\Gamma} \max_{\text{im}\Gamma} F$ contradicting the definition of c as the inf of F , exactly in the same way as in the proof of the Mountain Pass Theorem following from the classical Deformation Lemma using only the first derivative and the deformation towards the gradient.

2 Examples in finite dimensional setting

Consider $F(x) = x^3 + x^5$, then $F'(x) = 3x^2 + 5x^4$, $F''(0) = 0$ and $F'''(0) = 6$ and thus the solvability of $x^3 + x^5 = y$ can be derived from the solvability of $6h^3 = y$ which is trivial and the solution is given by $h = (y/6)^{1/3}$. The Palais–Smale condition holds for any fixed y since $|x^3 + x^5 + y|^2$ is bounded with respect to x iff $|x|$ is bounded.

Let us move to higher dimensions and consider $F(x) = (x_1^3 + x_1^5 - x_2^5, x_2^3 + x_2^5 + x_1^5)$. Then $F'_1(x) = (3x_1^2 + 5x_1^4, -5x_2^4)$ and $F'_2(x) = (5x_1^4, 3x_2^2 + 5x_2^4)$. Next $F'''(0)h^3 = 6(h_1^3, h_2^3)$. The Palais–Smale condition holds in this case since $(x_1^3 + x_1^5 - x_2^5 + y_1)^2 + (x_2^3 + x_1^5 + x_2^5 + y_2)^2$ is coercive with respect to $|x|$ growing like $x_1^{10} + x_2^{10}$ for any y .

3 Applications to integral or differential equations

Consider the problem suggested by Fijałkowski

$$F(x) = A(x^2)x + r(x) \tag{3.1}$$

where for some measurable function $0 < \alpha \leq K(t, s) \leq \beta < \infty$ the integral operator A is defined as follows

$$Az(t) = \int_0^1 K(t, s)z(s)ds. \tag{3.2}$$

Then $X = H = L^2(0, 1)$, while $A : L^1(0, 1) \rightarrow C([0, 1])$ is continuous and compact. If we assume $r(x) = o(|x|^3)$ at 0 then

$$F^{(k)}(0) = 0$$

for any $k = 1, 2$ while

$$F^{(3)}(0)h^3 = 6A(h^2)h.$$

Notice that, due to the lower bound for the function K and the growth assumption $o(|x|^3)$ at ∞ on r , the functional $|F(x)|^2$ is coercive in $x \in L^2(0, 1)$. Next due to the reflexivity of the space of the square integrable functions the Palais–Smale sequence contains the weakly convergent subsequence x_n in $L^2(0, 1)$ not to zero (otherwise we are done) and Ax_n^2 can be assumed to be convergent in $C([0, 1])$. On the function r we have to impose the assumption of weak continuity together with its derivative to get the convergence of $r(x_n)$ and $r'(x_n)$. Since

$$x_n = (F(x_n) - r(x_n))/(Ax_n^2) \quad (3.3)$$

and using the estimate $Ax_n^2(t) \geq \alpha|x_n|^2 > 0$ one gets that the convergence of $F(x_n)$ is equivalent to x_n under weak continuity assumption imposed on the function r . Finally recall that for the Palais smale sequence we have as $n \rightarrow \infty$ that

$$\langle F'(x_n), F(x_n) - y \rangle \rightarrow 0.$$

But using (3.3) one obtains the relation guaranteeing the convergence of $F(x_n)$

$$F'(x_n)h = A(x_n^2)h + 2A(x_nh)x_n + r'(x_n)h, \quad (3.4)$$

$$F'(x_n)h = A(x_n^2)h + 2A(x_nh)(F(x_n) - r(x_n))/(Ax_n^2) + r'(x_n)h. \quad (3.5)$$

4 Final comments and remarks

Local invertibility in non-smooth setting was considered among others by Clarke in [5] and Pales in [12], while set-valued mappings were considered in the context of local invertibility by Frankowska in [7]. The structure of mountain pass level was analyzed in the paper of Pucci and Serrin [13]. The implicit function theorem in the guide similar to [8] was treated by Idczak in [9]. One can formulate appropriate counterpart for functions with degenerate critical points as in Thm 1.1. Some applications of global diffeomorphism and global implicit function theorems to Hammerstein [2], Urysohn [3] and Volterra [4] integral equations should also be mentioned.

5 Appendix

Recall the following deformation lemma presented in the book of Jabri [10] originally proved by Shafrir.

Theorem 5.1 [Deformation Lemma, Shafrir [14]] *Let F be a C^1 -functional defined on a Banach space X and let $A \subset X$ be a closed set. Then, there exists a continuous deformation $\eta : [0, 1] \times X \rightarrow X$ satisfying*

- $\eta(0, x) = x$ for all $x \in X$,

- $\eta(t, x) = x$ for all $t \in [0, 1]$ if $x \in A$ or $F'(x) = 0$,
- $F(\eta(t, x)) \leq F(x)$ for all $t \in [0, 1]$ and $x \in X$,
- $F(\eta(t, x)) < F(x)$ for all $t \in (0, 1]$ if $x \in X \setminus A$ and $F'(x) \neq 0$.

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